EXTENSION OF SOME RESULTS FROM NETWORKS TO COVERS

A Master's Thesis

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EXTENSION OF SOME RESULTS FROM NETWORKS TO COVERS

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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In this thesis, we use covers as an extension of networks. The cover notion is almost the same as the conference structure that is proposed by Myerson. However, we extend several notions pertaining to networks to covers in different ways, reflecting the differences in our points of departure. In this framework, we extend the result of Jackson and Wollinsky(1996) to covers which provides a characterization of the Shapley-Myerson allocation rule in terms of component balancedness and equal bargaining power in networks.

Keywords: Networks, Covers, Component Balancedness, Equal Bargaining Power, Shapley-Myerson Allocation Rule.

ÖZET

AĞLARDAKİ BAZI SONUÇLARIN ÖRTÜLERE GENİŞLETİLMESİ

PER, Emre

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Prof. Semih Koray

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Bu tezde, örtüleri ağların genelleştirmesi olarak kullandık. Örtü kavramı, Myerson'ın tanımladığı konferans yapısı ile büyük oranda aynı şeyi ifade etmektedir. Fakat, ağlar için varolan bazı kavramları örtüler için genelleştirirken daha önce Myerson'in konferans yapılarında takip ettiği yaklaşımdan daha farklı bir bakış açısı izledik. Bu çerçevede, Jackson ve Wollinsky'nin bileşen-dengelilik ve eşit pazarlık gücü açısından Shapley-Myerson dağıtım kuralı için elde ettikleri tanımlandırma sonucunu örtüler için genelleştirdik.

Anahtar Kelimeler: Ağlar, Örtüler, Bileşen-dengelilik, Eşit Pazarlık Gücü, Shapley-Myerson Dağıtım Kuralı.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Founders of modern game theory John von Neumann and Oskar Morgernstern stated that their objective is to find "a mathematically complete principle to explain the rational behaviour of participants in a social economy and analyze the characteristics of that behaviour." Their seminal book (Von Neumann and Morgernstern 1944) laid the foundations of cooperative game theory. After von Neumann-Morgernstern solution concept, Shapley constructed a new solution concept in his Ph.D. dissertation (Shapley 1953). This solution requires that each player receive a weighted average of his marginal contribution to various coalitions. In cooperative game theory, the notion of coalition is a central one. However, only "coalitions" with one player are allowed in noncooperative game theory. Indeed, noncooperative games can be thought as a special form of cooperative games. When one deals with coalitions, the main questions are how the societal value is distributed among coalitions and how the players of the same coalition share their coalition's total payoff among themselves. Modern graph theory provides useful tools that allow to model and analyze cooperation structures in an attempt to answer these questions.

A graph can be thought of as a subset of the power set of vertices consisting of singleton and doubleton sets only whose union is equal to the entire vertex set such that no singleton is a subset of a doubleton. With such a representation of a graph, the notion of "cover" arises as a very natural generalization of a graph: namely, as any subcollection of the power set the union of whose members covers the vertex set provided that no set in the collection is a proper subset of another. Thus, this extension allows hyperedges of different orders in contrast to edges in a graph regarded as doubleton sets. On the other hand, there are many different scenarios in economic and social life, which seem to be compatible with covers. One possible scenario would be to consider a situation where a country may decide to join different international unions simultaneously which may have different sizes. The notion of a cover allows to capture the size differences as well as possible overlappings between different unions in a more adequate way than the notion of a network. But what we do in this study mainly consists of testing the robustness of certain results in network theory when these are extended to their counterparts in terms of covers.

The results we obtain seem to be promising in the sense that they provide hints for the nonvacousness of the extension in question. There is another companion strand of literature about conference structures[1] introduced by Myerson. Although the two notions are very similar, the differences in the points of departure that have separately and independently led to these notions are also reflected in the different ways used to extend certain other notions from networks to covers and conference structures, respectively. Thus, the two approaches give rise to different results.

We generalize some results concerning networks to covers, that is, to generalized networks where agents can form hyperlinks that can contain more than two players. However, covers and Myerson's conference structures are coincident in this context. But the fairness concepts and the restrictions of value functions that we use in defining the Shapley-Myerson allocation rule differ from those utilised by Myerson.

[1]We are grateful to Prof. Ferhad Hüseyin for having brought this literature to our awareness.

When the results we obtain for covers are restricted to cooperation structure with at most two vertices for each hyperlink, these results get reduced to familiar results in network theory. We define efficiency and stability for covers and investigate the tension between these two notions. Furthermore, we express the fairness condition for covers in such a way that the Shapley Myerson allocation rule becomes the unique allocation rule that satisfies component balancedness and "equal threat power" representing our fairness notion.

Finally, we modify network formation games for covers by defining hyperlink formation games in which every player submits the set of hyperedges that they want to get constructed, and a hyperedge is formed by the unanimous consent of all members in this hyperedge.

CHAPTER 2

LITERATURE SURVEY

Game theory studies strategic interactions between agents and provides a formal modelling approach to social situations in which decision makers interact with other agents. There are two main branches of game theory: cooperative and noncooperative. Noncooperative game theory deals with how rational individuals interact with one another in an effort to achieve their own goals, leading to what one calls "strategic games". On the other hand, a cooperative game is a game where groups of players can form coalitions, whose members cooperate by coordinating their actions. Hence a cooperative game captures competition between coalitions of players, rather than between individual players.

In cooperative games with transferable utility, a "characteristic function" specifies the maximal total payoff that each coalition can guarantee for itself instead of assigning individual payoffs to each agent. The notion of a characteristic form game can be traced back to von Neumann and Morgenstern(1944), where it is assumed that a coalition C plays against its complementary coalition N/C, inducing a "two-person game". Now there are different models to derive coalitional values from normal form games, but not all games in characteristic function form can be derived from normal form games. Formally, a characteristic function form game with side payments (also known as a TU-game) is given as a pair (N, v), where N denotes a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function. The characteristic function form has also been extended so as to include cases where utility is not transferable.

The characteristic function form focuses on the worth of each coalition separately, ignoring externalities possibly caused by how agents outside the coalition are coalesced. In the partition function form, on the other hand the payoff of a coalition depends not only on its members, but also on how the rest of the players are partitioned (Thrall and Lucas 1963).

Of the two types of games, noncooperative games enable us to model situations in a more detailed way, thus also leading to possibly more exact results. Cooperative games, on the other hand, are informationwise coarser. Most research in game theory, in general, focuses on how groups of people interact.

Myerson (1977) defines cooperation structures to talk about who is cooperating with whom among the players and used graph theoretic ideas in order to analyze cooperation structures in games. He adapted the Shapley formula for graphs as an allocation rule which is now known as the Shapley-Myerson allocaton rule in the literature. Myerson uses graphs as cooperation structures in studying the games in characteristic function form. Games in more general graph function form are defined for the first time in this paper. Indeed, the passage from coalitions to cooperation structures via graphs is accomplished by the partition function form: A graph partitions the set of players into communicating groups through its component structure. Graph function form games are TU games in which the value of each coalition is defined as the sum of the values of its connected subcoalitions induced by the given graph with respect to the given cooperative game v. Myerson contends that the distribution of value should be based on the connectedness structure of the underlying graph. However, the particular architecture of the coalition dos not matter. In order to overcome this shortcoming, explicit value functions on the set of all networks are introduced by Jackson and Wollinsky(1996), whereby they modify Myerson's cooperation structures by using networks. They start with a value function rather than a characteristic function. We follow a similar approach as Jackson and Wollinsky(1996) in the context of covers. Thus, the value function is defined on the set of all covers at the outset. This allows us to investigate several aspects pertaining to covers in a more comprehensive way than Myerson's conference structures. Jackson and Wollinsky(1996) study the tension between the stability and efficency in networks. Jackson and Wollinsky also introduced the notion of pairwise stability which may not be so meaningful in the setting of coverings. A network is pairwise stable if no player would be better off if he severed one of his links, and no pair of players would both benefit with at least one of them in the strictly sense from adding a new link among themselves. This may be regarded as a rather weak stability notion for networks as well since it does not allow deviations where a player deletes more than one link at a time or add new links with others simultaneously.

Some stronger forms of stability were also defined following this notion. Especially, strong stability defined by Dutta and Mutuswami(1997) is a stability notion which takes into account the deviations of groups of players of any size.

Aumann and Myerson(1988) modeled network formation as an extensive form game. They defined a two stage game in which, according to an exogeneously given ranking of pairs of players, the pairs in turn decide whether or not to form a link knowing the decisions of all the pairs before them and forecasting the decisions of pairs following them. However, this extensive form is necessarily ad hoc and makes the game difficult to analyze beyond very simple examples. Morover, the ordering of links can have a non-trivial impact on which networks emerge. These problems prompted some other approaches on network formation. Myerson(1991) modeled network formation as a normal formal game where the players simultaneously propose the links that they want to form, and the links are formed under consent of the both sides. This game is simple and captures the idea of link formation, but it generally has a large multiplicity of Nash equilibria. So, some refinements of the Nash equilibrium concept such as pairwise Nash equilibrium, strong Nash equilibrium and coalition proof Nash equilibrium were introduced in order to overcome this multiplicity problem.

Furthermore, Myerson (1980) defined conference structures to describe how the outcome of a cooperative game might depend on which groups of players hold cooperative planning conferences. The results of this paper generalize Myerson(1977)'s results by dropping the side payments assumption and by allowing for conferences of more than two players.

In the next chapter, we firstly present formal definitions and some results from the network literature. Then, we summarize the formal model concerning graph function forms defined by Myerson. Finally we describe conference structures and related notions introduced by Myerson along with the relevant results. In Chapter 4, we define the notion of a cover and present our findings. Chapter 5 concludes the study.

CHAPTER 3

NETWORKS AS COOPERATION STRUCTURES AND CONFERENCE STRUCTURES

3.1 Networks

Let N be a nonempty and finite set. A *graph* on N is a set of unordered pairs of distinct members of N. We refer to these pairs as links or edges of g. We set $N = \{1, 2, ..., n\}$ and interpret it as the set of individuals in the society who form bilateral relations among themselves represented by the edges of a graph. To reflect this interpretation, we will henceforth refer to a graph on N as a network on N. Thus, in a network g, $\{i, j\} \in g$ where $i, j \in N$ with $i \neq j$ means that i and j are linked in the network g. We will simply write ij for the link $\{i, j\}$.

The network $g^N = \{ij \mid i, j \in N, i \neq j\}$ is the complete network on N. We write $\mathscr{G} = \{g \mid g \subseteq g^N\}$ and refer to $\mathscr{G}$ as the set of all networks on N.

The network obtained by adding link ij to a network g is denoted by $g+ij$, and the network obtained by deleting link ij from a network g is denoted by $g - ij$. Let $N(g)$ be the set of players who have at least one link in g, i.e., $N(g) = \{i \in N \mid \exists j \in N \quad \text{such that} \quad ij \in g\}$.

Definition. A *path* in a network $g \in \mathscr{G}$ between $i, j \in N$ is a sequence of players $i_1, i_2, ..., i_m$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, 2, ..., m-1\}$ with $i_1 = i$ and $i_m = j$.

A network $g \in \mathscr{G}$ is said to be *connected* if, for any $i, j \in N$, there exists a path between i and j that starts at i and ends at j and it also stays in g.

Definition. $g' \subset g$ is a *component* of a network g if

1. For any $i, j \in N(g')$ where $i \neq j$ there exists a path in g' between i and j, and
2. For any $i \in N(g')$, $j \in N$, if $ij \in g$, then $ij \in g'$.

Indeed, components are the maximal connected subgraphs. The set of components of g is denoted by $C(g)$.

A function $v : \mathscr{G} \rightarrow R$ is called a value function on $\mathscr{G}$.

A value function represents the total value that is generated by a given network. Note that a value function is more informative than a cooperative TU game since the total payoff assigned to a coalition T not only depends upon the members of T, but also the architecture joining them if a coalition in a network is regarded as the set of vertices in a component.

Definition. A value function v is *component additive* if

$$v(g) = \sum_{g' \in C(g)} v(g')$$

for any $g \in \mathscr{G}$, where $C(g)$ stands for the set of all components of g.

We denote the set of all value functions on $\mathscr{G}$ by $\mathscr{V}_{\mathscr{G}}$ and suppress the subscript $\mathscr{G}$ whenever it is self-understood.

An allocation rule specifies how the total value generated by a network is distributed among the members in the society.

Definition. An *allocation rule* is a function $Y : \mathscr{G} \times \mathscr{V} \to R^n$, for any value function $v \in \mathscr{V}_\mathscr{G}$ and for any $g \in \mathscr{G}$, $\sum_{i \in N} Y_i(g, v) = v(g)$ and $Y_i(g, v) = 0$ whenever $i \notin N(g)$.

Definition. An allocation rule Y is *component balanced* if, for any component additive value function v on $\mathscr{G}$, $g \in \mathscr{G}$ and $g' \in C(g)$,

$$\sum_{i \in N(g')} Y_i(g, v) = v(g')$$

Whenever $v \in \mathscr{V}$ is unambigiuous, we will write simply $Y_i(g)$ instead of $Y_i(g, v)$

Note that component balancedness requires that under a component additive value function, the value generated by any component be distributed to the players within that component.

Definition. An allocation rule Y satisfies *equal bargaining power* if, for any component additive value function $v \in \mathscr{V}_\mathscr{G}$, $g \in \mathscr{G}$ and $i, j \in N$ with $i \neq j$,

$$Y_i(g) - Y_i(g - ij) = Y_j(g) - Y_j(g - ij)$$

Note that equal bargaining power does not require that players split the marginal value of a link. Equal bargaining power provides that players equally benefit or suffer from the addition of the link between themselves.

3.2 Networks as Cooperation Structures

Myerson(1977) defines partition function form games since the characteristic function form games ignore the link architecture between the players outside a given coalition in determining its value.

We refer to a nonempty subset of N as a coalition in N.

Let $S \in 2^N / \{\emptyset\}, g \in \mathscr{G}$ and $i, j \in S$ be given

Then i and j are connected in S by g if and only if there is a path in g connecting i to j via vertices in S.

Given $g \in \mathscr{G}$ and $S \in 2^N/\{\emptyset\}$, there exists unique partition of S which groups players together if and only if they are connected in S by g.

We will denote this partition by S/g. Formally,

$$S/g = \{\{i \in S \mid i \text{ and } j \text{ are connected in } S \text{ by } g\} \mid j \in S\}$$

Example 1. $N = \{1,2,3,4,5\}$ and $g = \{12,13,23,35,15\}$ then for $S_1 = \{1,2,3\}$, $S_2 = \{1,4,5\}$ and $S_3 = \{1,2,4,5\}$

$S_1/g = \{1,2,3\}/g = \{\{1,2,3\}\}$, $S_2/g = \{1,4,5\}/g = \{\{1,5\},\{4\}\}$,

$S_3/g = \{1,2,4,5\}/g = \{\{1,2,5\},\{4\}\}$ and $N/g = \{1,2,3,4,5\}/g = \{\{1,2,3,5\},\{4\}\}$

The idea behind this notion is the understanding that even if two players do not have a direct link between themselves, they may still effectively cooperate if they are connected by the graph g.

Definition. A *Transferable Utility* game is defined as an ordered pair (N, v) where $N = \{1,2,...,n\}$ is the set of players and $v : 2^N \to R$ is a function such that $v(\emptyset) = 0$. We refer to v as a characteristic function and interpret $v(S)$ as the worth of coalition $S \in 2^N$ which the members of S are yet to divide among themselves. Given a game (N, v) and a subset of players S, (S, v) is the subgame obtained by restricting v to subsets of S only.

Set of all relationships are cooperation structures and non-directed graphs are used for the cooperation structures.

In partition form games, Myerson(1977) defines the fair allocation rule $Y : \mathscr{G} \to R^n$ such that for any $g \in \mathscr{G}$ and $ij \in g$,

$$Y_i(g) - Y_i(g - ij) = Y_j(g) - Y_j(g - ij).$$

Indeed, the fair allocation for the characteristic function form game and the equal bargaining power for the network games coincide. Because the games are identical and definitions of these fairness conditions are the same although they have different names. The only difference is that second one uses the value functions defined over the set of all networks rather than using the characteristic function.

If we require that players can only communicate along links in g, then for any characteristic function game v and a graph g, define v/g to be a characteristic function game so that for any coalition $S \in 2^N/\{\emptyset\}$,

$$(v/g)(S) = \sum_{T \in S/g} v(T).$$

Note that $v/g^N = v$.

Shapley value is one of the most popular solution concepts in cooperative game theory. While allocating the payoffs, it considers every player's marginal contributions to possible coalitions.

Definition. The *Shapley value operator* is defined as

$$\varphi_i(v/g) = \sum_{S \subset N/\{i\}} [v/g|_{S \cup \{i\}} - v/g|_S].\frac{|S|!(n - |S| - 1)!}{n!}$$

for any $g \in \mathscr{G}$ and $i \in N$.

Thus, each player gets the weighted sum of his/her contributions to every possible coalition. In the literature of network games, this formula for the networks is named as Shapley-Myerson allocation rule.

Myerson (1977) proved that for any characteristic function game v, there is a unique fair allocation rule $Y : \mathscr{G} \rightarrow R^n$ satisfying component balancedness(he defined this as an efficiency notion) and fair allocation rule. This allocation rule also satisfies

$$Y(g) = \varphi(v/g)$$

where $\varphi(.)$ is the Shapley value operator.

Jackson and Wollinsky (1996) extends this result to component additive value functions for the networks by showing that the unique allocation rule which satisfies component balancedness and equal bargaining power is the Shapley Myerson allocation rule.

Myerson(1977) extends his ideas to non-transferable utility games and partition function form games. Moreover, in order to show the full generality of his ideas, he introduces the games in graph function form.

3.3 Games in Graph Function Form

An *embedded subgraph* is a pair (S, g) such that g is a graph and S is a connected component of g in N.

$ESG = \{(S, g) \mid g \in \mathscr{G}, S \in N/g\}$ where ESG represents the set of all embedded subgraphs.

A set $W \subseteq R^S$ is *comprehensive* if and only if for any $a \in W$ and $b \in R^S$

$$if \quad a_n \geq b_n \quad \forall n \in N, \quad then \quad b \in W.$$

W is a proper subset of R^S if and only if $W \in R^S$ and $\emptyset \neq W \neq R^S$.

Let ∂ denote the boundary operator, so that if $W \in R^S$ then ∂W is the boundary of W in R^S.

A graph function form game is a set valued function $w(.)$ with domain ESG, such that for any $(S, g) \in ESG$, $w(S, g)$ is a closed and comprehensive proper subset of R^S. The set of utility allocations which are feasible for the players in S when g is the set of bilateral cooperation links is denoted by $w(S,g)$. If cooperation structure g is given, then $\partial w(S,g)$ is the Pareto optimal frontier for the members of S.

A characteristic function form game v can be identified with a graph function form game $w(.)$ if and only if $w(S,g) = \{r \in R^S \mid \sum_{n \in S} r_n \leq v_S\}$ for all $(S,g) \in ESG$.

3.4 Conference Structures

Myerson (1980) introduced the conference structures in order to describe how the outcome of cooperative game might depend on which groups of players conduct cooperative planning conferences. Similar to the networks, allocation rules assign each conference structure to an allocation. Although Myerson did not describe the internal structure of these conferences, he generalized his 1977 paper's results by dropping side payments assumption and by allowing conference consisting of more than two players.

We will formally give the definitions of Myerson(1980) and summarize his results.

Let V be a characteristic function game without side payments and as usual N represents the nonempty and finite set of players in V where $N = \{1, 2, ..., n\}$. V maps each set of players $S \in 2^N/\{\emptyset\}$ to a subset of R^N where for any $S \in 2^N/\{\emptyset\}$,

i. $V(S)$ is a closed proper subset of R^N

ii. For any $x \in V(S), y \in R^n$ if for any $i \in S$, $y_i \leq x_i$ then $y \in V(S)$

This is somewhat weaker than the usual definition of a characteristic function game without side payments. We shall not need convexity of $V(S)$. However, we require $V(S)$ to be a comprehensive subset of R^n.

For any set $S \in 2^N/\{\emptyset\}$, let $\partial V(S)$ be the weakly Pareto efficient frontier of $V(S)$, that is $\partial V(S) = \{x \in V(S) | \quad \text{if} \quad y_i > x_i \quad \forall i \in S, \text{then} \quad y \notin V(S)\}$

Myerson aimed to describe the cooperation of players who will come together in a committee meeting or a conference in order to discuss their cooperative plans so that according to Myerson(1980), a *conference* represents any set of two or more players who might meet together to discuss their cooperative plans and a *conference structure* is defined as any collection of conferences.

Let CS denote the set of all possible conference structures. That is given formally as: $CS = \{Q \subset 2^N/\{\emptyset\} \mid \forall S \in Q,\ S \subseteq N \quad \text{and} \quad |S| \geq 2\}$.

Players i and j can cooperate even if they do not attend the same conference but it is the case only if there is a sequence of conferences $(S_1, ..., S_m)$ such that $i \in S_1$, $j \in S_m$ and $S_k \cap S_{k+1} \neq \emptyset$ for each $k \in \{1, 2, ..., m-1\}$.

If the above condition is satisfied then i and j are connected. Thus, they can be coordinated either by meeting together in some conference (This is the case when $m = 1$) or by attending distinct conferences that have some common members (This is the case when $m = 2$) or by some longer sequence of overlapping conferences (This is the case when $m > 3$).

Similar to the network structures, a component is defined as a maximal connected subset of the given conference structure. These maximal connected subsets represent the coalitions that can cooperate. Myerson did not name these as components but we use the term component for simplicity.

The components are refered as the set of maximal connected coalitions of conference structure Q and denoted by N/Q or it can be thought as the partitions of N defined by the connectedness relation in Q. Formally, N/Q can be written formally as:

$$N/Q = \{\{j \in N \mid \text{i and j are connected by Q}\} \mid i \in N\}$$

For any given conference structure $Q \in CS$, any conference $S \subseteq N$ and any individual $i \in N$, $Q - S$ is the conference structure differing from Q since

S is deleted from the list of conferences. $Q - S$ is formally defined as:

$$Q - S = \{T \mid T \in Q \quad \text{and} \quad T \neq S\}$$

$Q \cap^* S$ contains the conferences that can involve members only from S. Thus, any conference that contains players outside S is deleted from the list of conferences Q. $Q \cap^* S$ is formally defined as:

$$Q \cap^* S = \{T \mid T \in Q \quad \text{and} \quad T \subseteq S\}$$

$Q -^* i$ is the conference structure differing from Q in that all conferences that involve the player i are deleted. $Q -^* i$ is formally defined as:

$$Q -^* i = \{T \mid T \in Q \quad \text{and} \quad i \notin T\}$$

The outcome of the game V is expected to depend on how the players organize their conferences. Thus, each player's payoff can be given as a function of the conference structures. Allocation rules formally express this situation by assigning each conference structure to an allocation $X(Q)$ where $X(Q) = (X_1(Q), ..., X_n(Q))$. An allocation rule is formally defined for the game V as, $X : CS \to R^n$ such that for any conference structure $Q \in CS$, and for any conference $S \in N/Q$,

$$X(Q) \in \partial V(S)$$

where ∂ denotes the Pareto optimal frontier of V.

The above definition asserts that, if S is the maximal connected coalition for the conference structure Q, then the members of S cooperate in order to achieve a Pareto efficient allocation among those allocations available to them.

There are infinitely many points in $\partial V(S)$ and any of them can be chosen for the allocation. To obtain narrower range of interesting allocation rules, some additional restrictions can be set on the allocation rule.

While people cooperate with each other, they wish to obtain same additional benefit from their cooperation. This idea is the intuition behind the equal gains principle of Myerson(1980). This notion was named as equal bargaining power in network context (Jackson and Wollinsky, 1996). The counterpart of equal gains principle for the allocation rules in conference structures is defined as fair allocation rules (Myerson 1980).

Definition. An allocation rule $X : CS \to R^N$ is *fair* if and only if for any conference structure $Q \in CS$, any conference $S \in Q$ and $i, j \in S$

$$X_i(Q) - X_i(Q - S) = X_j(Q) - X_j(Q - S)$$

Indeed, fair allocation rule provides that each conference gives equal benefits to its members. If the members of the conference S decides not to organize this meeting then conference structure would be $Q - S$ and under the fairness condition each member of the conference S should enjoy the same gain or same loss from this change.

Organization of each conference depends on the consent of its each members so any member of this conference can cause the cancellation of this conference by withdrawing his support. This idea can be criticized because most of the conferences can be conducted without some of the members. Myerson(1980) named this notion as balanced contributions and if any allocation rule satisfies balanced contributions then player j's contribution to i is always equal to player i's contribution to j in any conference structure.

Definition. An allocation rule $X : CS \to R^N$ has *balanced contributions* if and only if for any conference structure $Q \in CS$ and $i, j \in N$,

$$X_i(Q) - X_i(Q -^* j) = X_j(Q) - X_j(Q -^* i)$$

As we discussed before, Shapley value considers the marginal contribution of each player to every possible coalition while allocating the total value. This rule was originally defined for games with side payments. Myerson(1980) extends this notion as an allocation rule for conference structures.

Definition. An allocation rule $X : CS \to R^N$ satisfies the *Shapley formula* if and only if for any conference structure $Q \in CS$ and $i \in N$

$$X_i(Q) - X_i(\emptyset) = \sum_{i \in S \subseteq N} \frac{(|S|-1)!(n-|S|)!}{n!}[Z(Q \cap^* S) - Z(Q \cap^* S -^* i)]$$

where $Z(Q) = \sum_{j \in N} X_j(Q)$ for any $Q \in CS$.

According to this formula, the payoff that the allocation rule assigns to each player is the weighted average of his/her contributions to the players in smaller conference structures.

The main purpose of Myerson(1980) is to describe the fair allocation rule and characterize this rule for the conference structures. His main result that characterizes the fair allocation is the following theorem:

Theorem 1 (Myerson (1980))**.** *There exists a unique fair allocation rule for the game V. This allocation rule also has balanced contributions and satisfies the Shapley formula, and no other allocation rule for V satisfies either of these properties.*

CHAPTER 4

COVERS

The notion of a "cover" arises as a very natural generalization of a graph, since we define a cover as a subcollection of the power set of N such that the union of the members in this subcollection covers the vertex set provided that no set in the collection is a proper subset of another. Thus, this extension allows hyperedges of different orders in contrast to edges in a graph regarded as doubleton sets.

4.1 Definitions and Notation

Let N be a finite, nonempty set. We set $N = \{1, 2, ..., n\}$ and interpret it as the set of players.

We will define cover in a formal way and give some definitions, each of which will be a counterpart of an existing definition in network literature.

Definition. A subset of 2^N, C is said to be *a cover* for N if it satisfies the following two conditions:

i. $\bigcup_{S \in C} S = N$.

ii. $\nexists S,\, S' \in C : S \neq S'$ and $S \subseteq S'$

We will denote the set of all covers for N by $\mathscr{C}^N$.

Definition. Given a cover $C \in \mathscr{C}^N$, a member $S \in C$ is said to be *a hyperedge* of order t (written as $ordS = t$) if $|S| = t + 1$.

Definition. A cover $C \in \mathscr{C}^N$ is said to be *connected* if for any $a, b \in N$, there exist $E_1, E_2, ..., E_k \in C$ such that $a \in E_1$, $b \in E_k$ and for any $i \in \{1, 2, ..., k-1\}$ $E_i \cap E_{i+1} \neq \emptyset$.

Proposition 1. *For any $C \in \mathscr{C}^N$, if C is connected, then $n \leq (\sum_{S \in C}(ordS)) + 1$.*

Proof. Let us take an arbitrary connected cover $C \in \mathscr{C}^N$. So for any $a, b \in N$, there exist $E_1, E_2, ..., E_k \in C$ such that $a \in E_1, b \in E_k$ and $E_i \cap E_{i+1} \neq \emptyset$ for all $i \in \{1, 2, ..., k-1\}$.

Define C' as the minimal connected cover obtainable from C such that for any $a, b \in N$, there exist $E_1' \subset E_1, E_2' \subset E_2, ..., E_k' \subset E_k$ and $E_1', E_2', ..., E_k' \in C'$ such that $a \in E_1', b \in E_k'$ and $E_i' \cap E_{i+1}' = \{j\}$ for some $j \in N$ for all $i \in \{1, 2, ..., k-1\}$.

Take all hyperedges of C', namely they are $E_1', E_2', ..., E_m'$.

Set $E_1' \cap E_2' = \{j_1\}$, $E_2' \cap E_3' = \{j_2\}, ..., E_{m-1}' \cap E_m' = \{j_{m-1}\}$ where $j_1, ..., j_{m-1} \in N$.

Assume that $|E_1'| = t_1 + 1, |E_2'| = t_2 + 1, ..., |E_m'| = t_m + 1$ for some $t_1, t_2, ..., t_m \in N/\{0\}$.

Consider $E_i'/\{j_i\}$ for $i \in \{1, 2, ..., m-1\}$.

Note that $(E_i'/\{j_i\}) \cap E_{i+1}'/\{j_{i+1}\} = \emptyset$ for $i \in \{1, 2, ..., m-2\}$.

Notice that $C'' = \{E_1'/\{j_1\}, E_2'/\{j_2\}, ..., E_{m-1}'/\{j_{m-1}\}, E_m'\}$ be a cover for N so that $n \leq |E_m'| + \sum_{i=1}^{m-1} |E_i'/\{j_i\}|$.

Indeed, $|E_m'| + \sum_{i=1}^{m-1} |E_i'/\{j_i\}| = t_m + 1 + (t_1 + t_2 + ... + t_{m-1})$

$$
\begin{aligned}
&= 1 + \textstyle\sum_{i=1}^{m} t_i \\
&= 1 + \textstyle\sum_{i=1}^{m}(|E_i'| - 1) \\
&= 1 + \textstyle\sum_{i=1}^{m}(ordE_i')
\end{aligned}
$$

Recall that $E_1^{'}, E_2^{'}, ..., E_m^{'}$ are all hyperedges of the minimal connected cover $C^{'}$ obtainable from C so $\sum_{i=1}^{m}(ordE_i^{'}) \leq (\sum_{S \in C}(ordS))$.

Thus, $1 + \sum_{i=1}^{m}(ordE_i^{'}) \leq 1 + (\sum_{S \in C}(ordS))$.

Hence $n \leq 1 + (\sum_{S \in C}(ordS))$. $\square$

There can be different ways to prove this inequality. For instance, one can prove it easily by induction on n.

The value of a cover is assigned by the value functions which can be any function that maps each cover to a real number. The value is the aggregation of individual payoffs. An allocation rule is defined with respect to the given value function in order to describe how the value associated with each cover is allocated among the individual players.

Definition. A function $v : \mathscr{C}^N \to R$ is called *a value function* for $\mathscr{C}^N$ if $v(C) = 0$ whenever $ordS = 0$ for all $S \in C$.

Definition. Given a value function $v : \mathscr{C}^N \to R$, a function $Y : \mathscr{C}^N \to R^N$ is called *an allocation rule* associated with v if, for any $C \in \mathscr{C}^N$, one has $v(C) = \sum_{i \in N}(Y_i(C))$.

4.2 Stability and Efficiency

Let $v : \mathscr{C}^N \to R$ be a value function and Y an allocation rule associated with v.

Definition. For any $C \in \mathscr{C}^N$, we say that C is *efficient relative to* v if

$$v(C) = \max_{C' \in \mathscr{C}^N} v(C')$$

Moreover, C is said to be *Pareto efficient relative to* (v, Y) if there is no $C' \in \mathscr{C}^N$ such that for any $i \in N : Y_i(C') \geq Y_i(C)$ and there is $j \in N : Y_j(C') > Y_j(C)$.

Efficiency indicates the maximal total value. Efficiency and Pareto efficiency are equivalent if the value is transferable across players. However, we are not dealing with transferable payoff games. In our case, the following result states the connection between these two efficiency conditions.

Proposition 2. *Let $v : \mathscr{C}^N \to R$ be a value function and $C \in \mathscr{C}^N$. C is efficient relative to v if and only if C is Pareto efficient relative to (v, Y) for any allocation rule Y associated with v.*

Proof. First assume that C is efficient relative to v. Take any allocation rule Y associated with v. Suppose that C is not Pareto efficient relative to v and Y. Then $\exists C' \in \mathscr{C}^N$ such that $\forall i \in N : Y_i(C') \geq Y_i(C)$ and $\exists j \in N : Y_j(C') > Y_j(C)$.

But then $v(C') = \sum_{i \in N} Y_i(C') > \sum_{i \in N} Y_i(C) = v(C)$ contradicts that C was efficient relative to v.

Conversely assume that C is Pareto efficient relative to v and Y for any allocation rule Y associated with v. Consider the egalitarian allocation rule Y_v^e (associated with v) where $Y_{v,i}^e(C) = \frac{v(C)}{|N|}$ for all $i \in N$

In particular, $C \in PE(v, Y_v^e)$

Take any cover $C' \in \mathscr{C}^N$.

Suppose that $v(C') > v(C)$. But then $\forall i \in N : Y_{v,i}^e(C') = \frac{v(C')}{|N|} > \frac{v(C)}{|N|} = Y_{v,i}^e(C)$ contradicts that $C \in PE(v, Y_v^e)$.

So, $\forall C' \in \mathscr{C}^N : v(C) \geq v(C')$.

Thus, $C \in E(v)$. $\square$

The result that we establish here is the counterpart of the one found for the networks which is shown almost in the same manner.

We want to define stability notions for covers. In order to do this, we should describe how coalitions of players behave and change their strategies. Namely, collection of T-function associated with a coalition T represents the ability of that coalition to change the cover.

Definition. Let $C \in \mathscr{C}^N$ and $T \in 2^N/\{\emptyset\}$. A function $f : C \to 2^N/\{\emptyset\}$ is called a *T-function on* C if $\forall S \in C : f(S) \subset S$ with $S/f(S) \subset T$. A cover $C' \in \mathscr{C}^N$ is said to be *obtainable from* C via T if $C' \subset \{f(S)|S \in C\} \cup 2^T$ for some T-function f on C.

Definition. Let $C \in \mathscr{C}^N$ and $k \in \{1, 2, ..., n\}$. We say that C is *k-stable relative to* (v, Y) if there is no $T \in 2^N/\{\emptyset\}$ with $|T| \leq k$ such that $\exists C' \in \mathscr{C}^N$ obtainable from C via T with $\forall i \in T : Y_i(C') \geq Y_i(C) \quad \exists j \in T : Y_j(C') > Y_j(C)$. C is said to be *strongly stable relative to (v, Y)* if C is k-stable relative to (v, Y) for all $k \in \{1, 2, ..., n\}$.

4.3 Shapley Myerson Allocation Rule

Definition. For any $C \in \mathscr{C}^N$ and $T \in 2^N/\{\emptyset\}$, we define the *restriction* of C to T, denoted as $C|_T$, by

$$C|_T = \{S \cap T | S \in C, \quad S \cap T \neq \emptyset \quad \text{and} \quad \nexists S' \in C : S \cap T \subset S' \cap T\} \cup \{\{i\} | i \in N/T\}$$

(Note that $C|_T \in \mathscr{C}^N$).

This restriction notion is necessary for us to define the Shapley-Myerson allocation rule. While characterizing the conference structures, Myerson followed a different way in defining the restriction and so in defining the Shapley Myerson allocation rule. He used $Q \cap^* S$ as the restriction of the conference structure over the coalition S. The conference structure $Q \cap^* S$ differs from the conference structure Q in that all conferences containing players outside S are eliminated. However, we define the restriction of cover C to coalition T by taking the nonempty intersections of coalition T with each hyperedge S. In our case, coalitions can continue to exist the hyeperedges without the players outside that coalition. On the other hand, in Myerson's definition if any conference involves players from that coalition, then that conference is canceled in the restriction.

Definition. *The Shapley-Myerson allocation rule* Y^{SM} associated with v is defined for any $C \in \mathscr{C}^N$ and $i \in N$ as

$$Y_i^{SM}(C) = \sum_{T \subset N/\{i\}} [v(C|_{T \cup \{i\}}) - v(C|_T)].\frac{|T|!(n-|T|-1)!}{n!}$$

We are looking for the existence of k-stable and strongly stable covers. For any given value funtion v, does there exist a k-stable cover $C \in \mathscr{C}^N$ relative to (v, Y^{SM})? We can answer this question for $k = 2$ by giving a counterexample in which we define a value funtion such that there does not exist a 2-stable cover.

There exists value functions under which there is no cover that is 2-stable with respect to Shapley-Myerson allocation rule. Following example shows this fact.

Example 2. $N = \{1, 2, 3\}, v(1, 2, 3) = 0, v(12, 3) = v(13, 2) = v(1, 23) = 10,$
$v(12, 23) = v(13, 23) = v(12, 13) = 11, v(12, 23, 13) = 12, v(123) = 13.$

Apply Shapley-Myerson allocation rule to allocate the total value.

$Y^{SM}(1, 2, 3) = (0, 0, 0)$

$Y^{SM}(12, 3) = (5, 5, 0), Y^{SM}(13, 2) = (5, 0, 5), Y^{SM}(1, 23) = (0, 5, 5).$

$Y^{SM}(12, 23) = (2, 7, 2), Y^{SM}(13, 23) = (2, 2, 7), Y^{SM}(12, 13) = (7, 2, 2).$

$Y^{SM}(12, 23, 13) = (4, 4, 4).$

$Y^{SM}(123) = (\frac{13}{3}, \frac{13}{3}, \frac{13}{3}).$

In this example, all covers except $C_1 = (1, 2, 3)$ are Pareto efficient relative to (v, Y^{SM}) and the unique efficient cover is $C_9 = (123)$.

Moreover, there are no 2-stable cover. Because for $C_1 = (1, 2, 3)$, any coalition with two players would lead to better payoffs for both of them. For $C_2 = (12, 3)$ Shapley Myerson give $(5, 5, 0)$; first and third players can cooperate and $C_2' = (13, 2)$ is obtainable from C_2 and in this time first player gets again 5 but third one gets also 5. So, C_2 is not 2-stable. By symmetry, $C_3 = (13, 2)$ and $C_4 = (1, 23)$ are not 2-stable. For $C_5 = (12, 23)$, Shapley Myerson give $(2, 7, 2)$; first and third players can cooperate and $C_5^{'} = (12, 23, 13)$ is obtainable from C_5 and in this time first player and third player both get 4 $(4 > 2)$. So, C_5 is not 2-stable. By symmetry $C_6 = (13, 23)$ and $C_7 = (12, 13)$ are not 2-stable. For $C_8 = (12, 23, 13)$, Shapley Myerson give $(4, 4, 4)$; first and third players can cooperate and $C_8^{'} = (13, 2)$ is obtainable from C_8 and in this time first player and third player both get 5 $(5 > 4)$. So, C_8 is not 2-stable. Similarly for $C_9 = (123)$ Shapley Myerson give $(\frac{13}{3}, \frac{13}{3}, \frac{13}{3})$; again first and third players can cooperate and $C_9^{'} = (13, 2)$ is obtainable from C_9 and in this time first player and third player both get 5 $(5 > \frac{13}{3})$. So, C_9 is not 2-stable. Thus, none of them is 2-stable.

For networks one can find for any given value function $v : G \to R$, there exists a network $g \in \mathscr{G}$ such that g is pairwise stable with respect to (v, Y^{SM}) (Jackson 2003a). However, we can not obtain a similar existence result for 2-stable covers. Even if we require the value function to be component additive and anonymous, the above counter example evaporates our hope to achieve such an existence result.

However, there always exists a 1-stable cover, because the cover in which all the players stay isolated give zero payoff to each player and no player can gain by deleting a link, because there is no link. So, this cover would automatically be 1-stable. In the following example, there is no 1-stable cover except the one in which all players are isolated.

Example 3. $N = \{1, 2, 3\}, v(1, 2, 3) = 0, v(12, 3) = v(13, 2) = v(1, 23) = -6$, $v(12, 23) = v(13, 23) = v(12, 13) = 0, v(12, 23, 13) = 6, v(123) = -6$.

Apply Shapley-Myerson allocation rule to allocate the total value.

$Y^{SM}(1, 2, 3) = (0, 0, 0), Y^{SM}(12, 3) = (-3, -3, 0), Y^{SM}(13, 2) = (-3, 0, -3)$, $Y^{SM}(1, 23) = (0, -3, -3), Y^{SM}(12, 23) = (3, -6, 3), Y^{SM}(13, 23) = (3, 3, -6)$,

$Y^{SM}(12,13) = (-6,3,3), Y^{SM}(12,23,13) = (2,2,2), Y^{SM}(123) = (-2,-2,-2)$.

Covers with one hyperedge are not 1-stable because players can delete the hyperedge and get zero rather than -3. Covers with two hyperedges are also not 1-stable since the player who has two hyperedges delete one of them so he gets -3 rather than -6. $(12,23,13)$ is not 1-stable since any of the player can delete one of its hyperedge and gets 3 rather than 2. (123) is not 1-stable because any player can choose being isolated and get 0 rather than -2.

Jackson and Wollinsky(1996) introduced the tension between stability and efficiency in networks and the literature grew after their work. This tension can also be investigated for covers. Following example is due to this problem.

Example 4. Let v be a component additive value function and we are given the Shapley Myerson allocation rule. $N = \{1,2,3\}, v(1,2,3) = 0, v(12,3) = \frac{7}{8}, v(13,2) = \frac{3}{8}, v(1,23) = \frac{3}{8}, v(12,23) = \frac{5}{8}, v(13,23) = 1, v(12,13) = \frac{5}{8}, v(12,23,13) = 1, v(123) = 1$.

$Y^{SM}(1,2,3) = (0,0,0)$, $Y^{SM}(12,3) = (\frac{7}{16}, \frac{7}{16}, 0)$, $Y^{SM}(13,2) = (\frac{3}{16}, 0, \frac{3}{16})$, $Y^{SM}(1,23) = (0, \frac{3}{8}, \frac{3}{8})$, $Y^{SM}(12,23) = (\frac{1}{4}, \frac{3}{8}, 0)$, $Y^{SM}(13,23) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, $Y^{SM}(12,13) = (\frac{3}{8}, \frac{1}{4}, 0)$, $Y^{SM}(12,23,13) = (\frac{3}{8}, \frac{3}{8}, \frac{1}{4})$, $Y^{SM}(123) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

As we consider this example, $(12,3)$ is the unique strongly stable cover which is Pareto efficient but not efficent. Morover, $(13,23)$, $(12,23,13)$ and (123) are the efficient covers but none of them is strongly stable, also none of them is $2-stable$ however all of them are 1-stable.

In partition function form, connectedness relation determines the coalition. Similar to the graph structures, we use connectedness relation in covers by taking components which are the maximal connected subcovers, the formal definition of a component of a cover is as follows:

Definition. A *component* of a cover C is a nonempty subcover $T \subset C$ such that

i. $\forall i, j \in N(T) : i \neq j, \quad \exists \{E_1, ..., E_k\} \subset T : i \in E_1, j \in E_k$ and $\forall t \in \{1, ..., k-1\}, \quad E_t \cap E_{t+1} \neq \emptyset$ (i.e. i and j are connected).

ii. $\forall i \in N(T), \forall S \in C : i \in S$, we have $S \in T$ holds.

Definition. A value function v is *component additive* if and only if for any cover $C \in \mathscr{C}^N$,

$v(C) = \sum_{T \in Cp(C)} v(C|_T)$

where $Cp(C)$ denotes the set of all components of a cover C.

Definition. An allocation rule Y is *component balanced* if for any component additive value function v, any cover $C \in \mathscr{C}^N$ and $T \in Cp(C)$

$$\sum_{i \in T} Y_i(C, v) = v(C|_T)$$

Component balancedness requires that the allocation rule distributes the resources generated by any component only to the members of that component.

Definition. A cover C is said to be *constrained efficient relative to* v if there is no cover $C' \in \mathscr{C}^N$ and a component balanced and anonymous allocation rule Y such that

$$\forall i \in N : Y_i(C') \geq Y_i(C) \quad \textit{and} \quad \exists j \in N : Y_j(C') > Y_j(C)$$

Definition. A value function v is *anonymous* if $v(C^\pi) = v(C)$ for any cover $C \in \mathscr{C}^N$ and permutation π

Definition. An allocation rule Y is *anonymous* if for any value function v, $C \in \mathscr{C}^N$ and permutation of agents, π, $Y_{\pi(i)}(C^\pi, v^\pi) = Y_i(C, v)$

The anonymity of an allocation rule means that the information used to decide on allocations is obtained from the value function and the particular cover and not from the labelling individuals. We use anonymity to give the relationship between the three efficiency notion that we defined. Let $E(v)$ and $CE(v)$ denote the efficient and constraint efficient covers with respect to the value function v, respectively and $PE(v, Y)$ denotes the Pareto efficient covers with respect to the value function v and allocation rule Y. The next result that we present is the counterpart of the one given in Jackson(2003a).

Proposition 3. *Let v be a component additive and anonymous value function. We have, $E(v) \subset CE(v) \subset PE(v, Y)$ for any component balanced and anonymous allocation rule Y associated with v.*

We will consider *Equal Bargaining Power* which is defined firstly by Myerson(1977) as a fairness condition for allocation rules in undirected graphs. Firstly, let us define the severing one hyperegde from the original cover for any cover $C \in \mathscr{C}^N$ and hyperedge $S \in C$ as:

$$C - S = C \setminus S \cup \{\{i\} | i \in S \quad and \quad \nexists S' \in C \setminus S : i \in S'\}$$

Definition. An allocation rule Y satisfies *Equal Bargaining Power* if for any component additive value function v, for any cover $C \in \mathscr{C}^N$, $S \in C$, and $i, j \in S$,

$$Y_i(C, v) - Y_i(C - S, v) = Y_j(C, v) - Y_j(C - S, v)$$

Indeed, we want to generalize one of the theorem of Myerson [which is firstly proven in his 1977 paper and extended in Jackson and Wolinsky(1996)] for covers. This theorem asserts that if v is component additive, then the unique allocation rule Y which satisfies component balancedness and equal bargaining power is the Shapley-Myerson allocation rule. However, this theorem is valid for networks. Firstly, we defined component balancedness and equal bargaining power and Shapley-Myerson allocation rule for covers. However, this theorem does not work for covers with its original form. We are going to show this with the following counter example.

Example 5. $N = \{1, 2, 3, 4\}$, anonymous value function v is defined as

$v(\{1,2,3\},\{1,4\}) = 15, v(\{1,2,3\},\{4\}) = 10, v(\{1,2\},\{3\},\{4\}) = 5,$

$v(\{1,2\},\{1,4\},\{3\}) = 10$ and the value function maps every other covers to zero. We are given the allocation rule as Shapley-Myerson allocation rule.

If we calculate the payoffs of each player under Shapley-Myerson allocation rule for each cover, we obtain:

$Y^{SM}(\{1,2,3\},\{4\}) = (\frac{10}{3}, \frac{10}{3}, \frac{10}{3}, 0)$

$Y^{SM}(\{1,2\},\{3\},\{4\}) = (\frac{5}{2}, \frac{5}{2}, 0, 0)$

$Y^{SM}(\{1,2\},\{1,4\},\{3\}) = (5, \frac{5}{2}, 0, \frac{5}{2})$

$Y^{SM}(\{1,2,3\},\{1,4\}) = (\frac{35}{6}, \frac{35}{12}, \frac{35}{12}, \frac{10}{3})$

Note that for every other cover (which does not have the same shape of one of these four covers) each player gets zero payoff.

Consider the cover C as $C = (\{1,2,3\},\{1,4\})$ and consider $C - \{1,4\}$

So, $Y^{SM}(C) = (\frac{35}{6}, \frac{35}{12}, \frac{35}{12}, \frac{10}{3})$ and $Y^{SM}(C - \{1,4\}) = (\frac{5}{2}, \frac{5}{2}, 0, 0)$

Then, $Y_1^{SM}(C) - Y_1^{SM}(C - \{1,4\}) \neq Y_4^{SM}(C) - Y_4^{SM}(C - \{1,4\})$ since

$Y_1^{SM}(C) - Y_1^{SM}(C - \{1,4\}) = \frac{35}{6} - \frac{10}{3} = \frac{15}{6}$ and $Y_4^{SM}(C) - Y_4^{SM}(C - \{1,4\} = \frac{10}{3} - 0 = \frac{10}{3}$.

Thus, we can not directly generalize the existence and uniqueness theorem for covers. So, we would define a new fairness condition which is called Equal Threat Power. In this case, we change the players' possible threats among themselves. If any player does not want to attend any hyperedge (or conference as Myerson named) then this hyperedge is preserved without this player (in conference notation, the conference is not canceled and done without this player). In balanced contribution approach, any player has a right to cancel any conference of which he is a member. Our approach and Myerson's balanced contribution approach can be compared and discussed according to the situation that is desired to be modeled.

We define $C_{i,j}$ as the cover that is constructed as the threat of the player i for the player j.

Definition. For any $i, j \in N$ such that $i \neq j$, the threat of the player i for the player j, $C_{i,j}$ is given as

$$C_{i,j} = \begin{cases} C & \text{if} \quad \nexists S \in C \text{ s.t. } i,j \in S \\ \{S \in C : i \notin S\} \cup \{S' \in 2^N/\{\emptyset\} : S' \cup \{i\} \in C\} & \text{otherwise} \end{cases}$$

Definition. An allocation rule Y satisfies *Equal Threat Power* if for any component additive value function v, for any cover $C \in \mathscr{C}^N$, $S \in C$, and $i, j \in S$,

$$Y_{v,i}(C) - Y_{v,i}(C_{j,i}) = Y_{v,j}(C) - Y_{v,j}(C_{i,j})$$

Theorem 2. *For any component additive value function v, the unique allocation rule Y which satisfies component balancedness and equal threat power is the Shapley-Myerson allocation rule.*

Proof. Firstly we show that there can be at most one allocation rule for a given game v that satisfies equal threat power and component balancedness. Indeed, suppose $Y^1 : \mathscr{C}^N \rightarrow R^N$ and $Y^2 : \mathscr{C}^N \rightarrow R^N$ both satisfy the component balancedness and equal threat power and $Y^1 \neq Y^2$.

Let C be a cover with a minimum number of hyperedges such that $Y^1 \neq Y^2$, set $y^1 = Y^1(C)$ and $y^2 = Y^2(C)$ so that $y^1 \neq y^2$.

By the minimality of C, if $C_{i,j}$ is the new cover in which some players leave their hyperedges, then $Y^1(C_{i,j}) = Y^2(C_{i,j})$.

Thus, $\forall i, j \in S : Y_i^1(C_{j,i}) = Y_i^2(C_{j,i})$ and $Y_j^1(C_{i,j}) = Y_j^2(C_{i,j})$

If we substract the last two equalities from each other, we obtain

$$Y_i^1(C_{j,i}) - Y_j^1(C_{i,j}) = Y_i^2(C_{j,i}) - Y_j^2(C_{i,j}) \qquad (1)$$

If we consider the equal threat power:

$$Y_i(C) - Y_i(C_{j,i}) = Y_j(C) - Y_j(C_{i,j}) \Rightarrow Y_i(C) - Y_j(C) = Y_i(C_{j,i}) - Y_j(C_{i,j}) \quad (2)$$

Write the equality (2) for both Y^1 and Y^2

$$Y_i^1(C) - Y_j^1(C) = Y_i^1(C_{j,i}) - Y_j^1(C_{i,j})$$

$$Y_i^2(C) - Y_j^2(C) = Y_i^2(C_{j,i}) - Y_j^2(C_{i,j})$$

By equation (1), we know that the right hand side of the last two equations are equal.

So, $Y_i^1(C) - Y_j^1(C) = Y_i^2(C) - Y_j^2(C)$

$\Rightarrow y_i^1 - y_j^1 = y_i^2 - y_j^2$

$\Rightarrow y_i^1 - y_i^2 = y_j^1 - y_j^2$

So, $y_i^1 - y_i^2 = y_j^1 - y_j^2$ whenever i and j are in the same hyperedge. This means that $y_i^1 - y_i^2 = y_j^1 - y_j^2$ for each i and j which are in the same component.

By component balancedness of y^1 and y^2, we have $\sum_{n\in T} y_n^1 = \sum_{n\in T} y_n^2$ where T is any component of C.

But then $\sum_{n\in T}(y_n^1 - y_n^2) = 0$.

We showed that $y_i^1 - y_i^2 = y_j^1 - y_j^2$ whenever i and j are in the same component. Set $y_n^1 - y_n^2 = d_{T(C)}$ for any $n \in T$.

Then $0 = \sum_{n\in T}(y_n^1 - y_n^2) = |T|.\ d_{T(C)}$

Since $T > 0, d_{T(C)} = 0$. Thus $y_n^1 - y_n^2 = 0$ for any $n \in T$.

Hence $y^1 = y^2$, a contradiction.

That is, there can be at most one allocation rule for any given game v that satisfies component balancedness and equal threat power.

Claim. *I* *Y^{SM} satisfies component balancedness.*

Proof. Take any component $T \subset C$.

We want to show that for all $T \subset C$, $\sum_{i\in N(T)}(Y_i^{SM}(C|_T)) = v(C|_T)$. For simplicity, take $v(C|_T) = v(T)$

$N(T) = \{i \in N : i \in S \quad \text{for some hyperedge S and} \quad S \subset T\}$

Consider any $j \in N/N(T)$ where $N(T)$ denotes the set of players in the componenet T. Compute $Y_{v,j}^{SM}(T)$.

$Y_{v,j}^{SM}(T) = \sum_{S\subset N/\{j\}}[v(T|_{S\cup\{j\}}) - v(T|_S)].\frac{|S|!(n-|S|-1)!}{n!}$

$v(T|_{S\cup\{j\}}) - v(T|_S) = 0$ since j is singleton in T.

So, $v(T) = \sum_{i\in N} Y_{v,i}^{SM}(T) = \sum_{i\in N(T)} Y_{v,i}^{SM}(T)$ for all components T of C.

Hence, Y^{SM} satisfies component balancedness. $\square$

Claim. *II* *Y^{SM} satisfies equal threat power.*

Proof. Take any $C \in \mathscr{C}^N$ and any $i, j \in N$ such that $i \neq j$.

We want to show that

$Y^{SM}_{v,j}(C) - Y^{SM}_{v,j}(C_{i,j}) = Y^{SM}_{v,i}(C) - Y^{SM}_{v,i}(C_{j,i})$

Firstly, consider the case that i and j have no common hyperedge.

Then $C_{i,j} = C$ and $C_{j,i} = C$.

So, $Y^{SM}_{v,j}(C) - Y^{SM}_{v,j}(C_{i,j}) = Y^{SM}_{v,i}(C) - Y^{SM}_{v,i}(C_{j,i})$.

Suppose that i and j are found in the same hyperedge.

So, $Y^{SM}_{v,i}(C) - Y^{SM}_{v,i}(C_{j,i}) = \sum_{S \subset N/\{i\}} [v(C|_{S\cup\{i\}}) - v(C|_S)].\frac{|S|!(n-|S|-1)!}{n!} - \sum_{S \subset N/\{i\}} [v(C_{j,i}|_{S\cup\{i\}}) - v(C_{j,i}|_S)].\frac{|S|!(n-|S|-1)!}{n!}$

$= \sum_{S \subset N/\{i\}} [v(C|_{S\cup\{i\}}) - v(C_{j,i}|_{S\cup\{i\}}) - v(C|_S) + v(C_{j,i}|_S)].\frac{|S|!(n-|S|-1)!}{n!}$

If $j \notin S$ then $v(C|_{S\cup\{i\}}) = v(C_{j,i}|_{S\cup\{i\}})$ and $v(C|_S) = v(C_{j,i}|_S$

Then $Y^{SM}_{v,i}(C) - Y^{SM}_{v,i}(C_{j,i}) = \sum_{j \in S \subset N/\{i\}} [v(C|_{S\cup\{i\}}) - v(C_{j,i}|_{S\cup\{i\}}) - v(C|_S) + v(C_{j,i}|_S)].\frac{|S|!(n-|S|-1)!}{n!}$

By symmetry

$Y^{SM}_{v,j}(C) - Y^{SM}_{v,j}(C_{i,j}) = \sum_{i \in \bar{S} \subset N/\{j\}} [v(C|_{\bar{S}\cup\{j\}}) - v(C_{i,j}|_{\bar{S}\cup\{j\}}) - v(C|_{\bar{S}}) + v(C_{i,j}|_{\bar{S}})].\frac{|\bar{S}|!(n-|\bar{S}|-1)!}{n!}$

$\sum_{j \in S \subset N/\{i\}} v(C|_{S\cup\{i\}}) = \sum_{i \in \bar{S} \subset N/\{j\}} v(C|_{\bar{S}\cup\{j\}})$

$\sum_{j \in S \subset N/\{i\}} v(C_{j,i}|_{S\cup\{i\}}) = \sum_{i \in \bar{S} \subset N/\{j\}} v(C|_{\bar{S}})$

$\sum_{j \in S \subset N/\{i\}} v(C|_S) = \sum_{i \in \bar{S} \subset N/\{j\}} v(C_{i,j}|_{\bar{S}\cup\{j\}})$

$\sum_{j \in S \subset N/\{i\}} v(C_{j,i}|_S) = \sum_{i \in \bar{S} \subset N/\{j\}} v(C_{i,j}|_{\bar{S}})$

Thanks to these four equalities we obtain $\forall C \in \mathscr{C}^N, \forall S \in C, \forall i, j \in S:$

$Y^{SM}_{v,i}(C) - Y^{SM}_{v,i}(C_{j,i}) = Y^{SM}_{v,j}(C) - Y^{SM}_{v,j}(C_{i,j})$

Hence, Y^{SM} satisfies equal threat power. □

So, the proof is complete and we obtain that the unique allocation rule that satisfies componenet balancedness and equal threat power is the Shapley-Myerson allocation rule. □

4.4 Hyperlink Formation Game

One branch of the economics literature that has recently emerged is the study of the endogeneous formation of social networks by self interested agents. Different versions of link formation in societies where agents are fully aware of the shape of the social network they belong to and the impact of the network on their well-being are modeled. Myerson (1991) suggests a noncooperative link formation game in which agents independently announce which links they would like to see in the network. In this framework, some of the popular game theoretic equilibrium concepts can be used to model which networks will form. After Myerson, other economic designers defined different link formation games. We want to generalize the usual link formation game of Myerson to the covers. Namely, we will define the hyperlink formation game and use some game theoretic solution concepts to make predictions about which covers will form. Similar to the network formation game, the decisions of the players about whether they want to form a hyperlink or not can be modeled as a strategic form game. The value function and the allocation rule would constitute the payoff function of the strategic form game.

Let N be the set of players, set $N = \{1, 2, ..., n\}$ and $v : \mathscr{C}^N \rightarrow R$ be a value function and allocation rule Y associated with v be given. Also $2^N/\emptyset = \{\{1\}, \{2\}, ..., \{n\}, ..., \{1, 2, ..., n\}\}$ is labeled uniquely, thus $2^N/\emptyset$ is uniquely ordered that 1^{th} element of $2^N/\emptyset$ is $\{1\}$, 2^{nd} element of $2^N/\emptyset$ is $\{2\}$, $(n+1)^{th}$ element of $2^N/\emptyset$ is $\{1, 2\}$ etc. and $(2^N - 1)^{th}$ element of $2^N/\emptyset$ is $\{1, 2, ..., n\}$. We call T_j as the j^{th} element of $2^N/\emptyset$.

For each $i \in N$, the strategy set is an $(2^N - 1)$-tuple of 0 and 1,

$S_i = \{0, 1\}^{2^N - 1} = \{(s_{i1}, s_{i2}, ...s_{i2^N - 1}) \mid s_{ij} \in \{0, 1\}\}$ for all $i \in N$ and s_{ij} denotes the j^{th} coordinate of s_i. If $s_{ij} = 1$, player i wants to form the j^{th} hyperlink which means the j^{th} element of $2^N/\emptyset$.

Write $S = \prod_{i \in N} S_i$.

Given the strategy profile S, the function $c : S \rightarrow \mathscr{C}^N$ is defined as

$\forall s \in S : c(s)$ such that

$E(c(s)) = \{T_j \in 2^N/\emptyset \mid \prod_{i \in N(T)} s_{ij} = 1\}$ where $E(c(s))$ is the set of all hyperlinks of $c(s)$.

For each $i \in N$ and $s \in S$, define $u_i(s) = Y_i(c(s))$. The normal form game $H = (N, S, u)$ is called the *Hyperlink Formation Game for N associated with* (v, Y).

Definition. A strategy profile $s \in S$ is called *a Nash Equilibrium of H* if for any $i \in N$ and $s_i^{'} \in S_i, \quad u_i(s) \geq u_i(s_i^{'}, s_{-i})$

Now a cover $C \in \mathscr{C}^N$ is called *Nash Stable* relative to (v, Y) if there exists $s \in S$ such that s is a Nash equilibrium of H and $c(s) = C$.

The idea of Nash equilibrium is that no single player can benefit from unilaterally changing his or her move - a non-cooperative best-response equilibrium.

Definition. A strategy profile $s \in S$ is said to be *a Strong Nash Equilibrium of H* if there is no coalition $Cl \in 2^N - \{\emptyset\}$ such that there exists $s_{Cl}^{'} \in S_{Cl} = \prod_{j \in Cl} S_j$: for any $i \in Cl$,

$$u_i(s_{Cl}^{'}, s_{N/Cl}) \geq u_i(s)$$

with at least one of the inequalities being strict.

Moreover, a cover $C \in \mathscr{C}^N$ is called *Strongly Nash Stable* relative to (v, Y) if there exists $s \in S$ such that s is a strong Nash equilibrium of H and $c(s) = C$.

Considering bilateral link formation, one can define the weaker version of Strong Nash equilibrium, which is immune to deviations of coalitions involving at most 2 member:

Definition. A strategy profile $s \in S$ is said to be *a* 2*-stable Nash Equilibrium of* H if there is no coalition with $|Cl| \leq 2$ such that there exists $s'_{Cl} \in S_{Cl} = \prod_{j \in Cl} S_j$ such that for any $i \in Cl$,

$$u_i(s'_{Cl}, s_{N/Cl}) \geq u_i(s)$$

with at least one of the inequalities being strict.

Moreover, a cover $C \in \mathscr{C}^N$ is called *2-stable Nash Stable* relative to (v, Y) if there exists $s \in S$ such that s is a 2-stable Nash equilibrium of H and $c(s) = C$.

This notion was previously defined and used in some papers focusing on link formation game. Francis Bloch and Matthew O. Jackson(2005) named this notion as *Strong Nash equilibrium of order* 2. Indeed, this notion is firstly mentioned by Jackson and Wollinsky(1996) as a potential refinement of pairwise stability. In addition to this, Goyal and Vega-Redondo(2005) prefered to use *bilateral equilibria* and Slikker and van den Nouweland(2005) prefered to use *pair stability* rather than 2-stability. However, previously we defined k-stable covers for each $k \in \{1, 2, ..., n\}$ so we prefer to call it 2-stable Nash equilibrium.

Definition. Let $v : \mathscr{C}^N \rightarrow R$ be a value function and allocation rule Y associated with v be given. A cover $C \in \mathscr{C}^N$ is said to be *Pairwise Stable* relative to (v, Y) if

1. $\forall S \in C, \forall i \in C : Y_i(C) \geq Y_i(C - S)$
2. $\forall S \notin C,$ if $Y_i(C + S) > Y_i(C)$ for some $i \in S,$ then $\exists j \in S : Y_j(C + S) < Y_j(C)$

Definition. Let $v : \mathscr{C}^N \rightarrow R$ be a value function and allocation rule Y associated with v be given. Writing $H(v, Y) = (N, S, u)$ for hyperlink formation game as usual, we define $C \in \mathscr{C}^N$ to be *Pairwise Nash Stable* relative to

(v, Y) if there exists some Nash equilibrium s of $H(v, Y)$ with $c(s) = C$ and there does not exist any $S \notin C$ such that

$$Y_i(C + S) \geq Y_i(C) \quad \forall i \in S \text{ and } Y_i(C + S) > Y_i(C) \text{ for at least one } j \in S$$

CHAPTER 5

CONCLUSION

In this work, we used the notion of a cover as a generalization of network structure. In a cover, a hyperedge can consist of more than two players, whereas in networks an edge is formed only by two players.

The cover notion that we use turned out to be almost the same as the concept of a conference structure introduced by Myerson (1980). However, the counterparts of several network notions extended to covers in the present study do reflect the philosophical difference of our point of departure from that of Myerson's. The Shapley-Myerson allocation rule that we define in our framework is thus different, and it is characterized uniquely by component balancedness conjoined with "equal threat power". Moreover, we showed that in general there does not exist a 2-stable covering with respect to Shapley-Myerson allocation rule, where the 2-stability notion is a stability notion that can be thought of to be closest to that of pairwise stability for networks in our setting.

There are various instances where coverings can be used as appropriate tools for modeling purposes. For example, trade agreements between different groups of countries where overlappings are allowed can be modeled using covers. This study leads to more open problems than those already dealt with here. Almost every problem in network theory, including network formation games, is yet to be extended to covers.

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